CALCULUS

6th Edition

Single & Multivariable

Hughes-Hallett Gleason McCallum et al.

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PREFACE

Calculus is one of the greatest achievements of the human intellect. Inspired by problems in astronomy, Newton and Leibniz developed the ideas of calculus 300 years ago. Since then, each century has demonstrated the power of calculus to illuminate questions in mathematics, the physical sciences, engineering, and the social and biological sciences.

Calculus has been so successful both because its central theme—change—is pivotal to an analysis of the natural world and because of its extraordinary power to reduce complicated problems to simple procedures. Therein lies the danger in teaching calculus: it is possible to teach the subject as nothing but procedures—thereby losing sight of both the mathematics and of its practical value. This edition of *Calculus* continues our effort to promote courses in which understanding and computation reinforce each other.

Mathematical Thinking Supported by Theory and Modeling

The first stage in the development of mathematical thinking is the acquisition of a clear intuitive picture of the central ideas. In the next stage, the student learns to reason with the intuitive ideas in plain English. After this foundation has been laid, there is a choice of direction. All students benefit from both theory and modeling, but the balance may differ for different groups. Some students, such as mathematics majors, may prefer more theory, while others may prefer more modeling. For instructors wishing to emphasize the connection between calculus and other fields, the text includes:

- A variety of problems from the **physical sciences** and **engineering**.
- Examples from the **biological sciences** and **economics**.
- Models from the **health sciences** and of **population growth**.
- New problems on **sustainability**.
- New case studies on **medicine** by David E. Sloane, MD.

Origin of the Text

From the beginning, this textbook grew out of a community of mathematics instructors eager to find effective ways for students to learn calculus. This Sixth Edition of *Calculus* reflects the many voices of users at research universities, four-year colleges, community colleges, and secondary schools. Their input and that of our partner disciplines, engineering and the natural and social sciences, continue to shape our work.

Active Learning: Good Problems

As instructors ourselves, we know that interactive classrooms and well-crafted problems promote student learning. Since its inception, the hallmark of our text has been its innovative and engaging problems. These problems probe student understanding in ways often taken for granted. Praised for their creativity and variety, the influence of these problems has extended far beyond the users of our textbook.

The Sixth Edition continues this tradition. Under our approach, which we called the "Rule of Four," ideas are presented graphically, numerically, symbolically, and verbally, thereby encouraging students with a variety of learning styles to expand their knowledge. This edition expands the types of problems available:

- New **Strengthen Your Understanding** problems at the end of every section. These problems ask students to reflect on what they have learned by deciding "What is wrong?" with a statement and to "Give an example" of an idea.
- **ConcepTests** promote active learning in the classroom. These can be used with or without clickers (personal response systems), and have been shown to dramatically improve student learning. Available in a book or on the web at www.wiley.com/college/hughes-hallett.

- **Class Worksheets** allow instructors to engage students in individual or group class-work. Samples are available in the Instructor's Manual, and all are on the web at www.wiley.com/college/hughes-hallett.
- Updated **Data and Models**. For example, Section 11.7 follows the current debate on *Peak Oil Production*, underscoring the importance of mathematics in understanding the world's economic and social problems.
- **Projects** at the end of each chapter provide opportunities for a sustained investigation, often using skills from different parts of the course.
- **Drill Exercises** build student skill and confidence.
- **Online Problems** available in WileyPLUS or WeBWorK, for example. Many problems are randomized, providing students with expanded opportunities for practice with immediate feedback.

Symbolic Manipulation and Technology

To use calculus effectively, students need skill in both symbolic manipulation and the use of technology. The balance between the two may vary, depending on the needs of the students and the wishes of the instructor. The book is adaptable to many different combinations.

The book does not require any specific software or technology. It has been used with graphing calculators, graphing software, and computer algebra systems. Any technology with the ability to graph functions and perform numerical integration will suffice. Students are expected to use their own judgment to determine where technology is useful.

Content

This content represents our vision of how calculus can be taught. It is flexible enough to accommodate individual course needs and requirements. Topics can easily be added or deleted, or the order changed.

Changes to the text in the Sixth Edition are in italics. In all chapters, many new problems were added and others were updated.

Chapter 1: A Library of Functions

This chapter introduces all the elementary functions to be used in the book. Although the functions are probably familiar, the graphical, numerical, verbal, and modeling approach to them may be new. We introduce exponential functions at the earliest possible stage, since they are fundamental to the understanding of real-world processes. The chapter concludes with a section on limits, allowing for a discussion of continuity at a point and on an interval. The section on limits is flexible enough to allow for a brief introduction before derivatives or for a more extensive treatment.

Chapter 2: Key Concept: The Derivative

The purpose of this chapter is to give the student a practical understanding of the definition of the derivative and its interpretation as an instantaneous rate of change. The power rule is introduced; other rules are introduced in Chapter 3.

Chapter 3: Short-Cuts to Differentiation

The derivatives of all the functions in Chapter 1 are introduced, as well as the rules for differentiating products; quotients; and composite, inverse, hyperbolic, and implicitly defined functions.

Chapter 4: Using the Derivative

The aim of this chapter is to enable the student to use the derivative in solving problems, including optimization, graphing, rates, parametric equations, and indeterminate forms. It is not necessary to cover all the sections in this chapter.

To increase access to optimization, many sections of this chapter have been streamlined. Optimization and Modeling are now in Section 4.3, followed by Families of Functions and Modeling in Section 4.4. Upper and lower bounds have been moved to Section 4.2, and geometric optimization is now combined with Optimization and Modeling. Section 4.8 on Parametric Equations is linked to Appendix D, allowing discussion of velocity as a vector.

Chapter 5: Key Concept: The Definite Integral

The purpose of this chapter is to give the student a practical understanding of the definite integral as a limit of Riemann sums and to bring out the connection between the derivative and the definite integral in the Fundamental Theorem of Calculus.

Section 5.3 now includes the application of the Fundamental Theorem of Calculus to the computation of definite integrals. The use of integrals to find averages is now in Section 5.4.

Chapter 6: Constructing Antiderivatives

This chapter focuses on going backward from a derivative to the original function, first graphically and numerically, then analytically. It introduces the Second Fundamental Theorem of Calculus and the concept of a differential equation.

Section 6.3 on Differential Equations and Motion contains the material from the former Section 6.5.

Chapter 7: Integration

This chapter includes several techniques of integration, including substitution, parts, partial fractions, and trigonometric substitutions; others are included in the table of integrals. There are discussions of numerical methods and of improper integrals.

Section 7.4 now includes the use of triangles to help students visualize a trigonometric substitution. The two former sections on numerical methods have been combined into Section 7.5.

Chapter 8: Using the Definite Integral

This chapter emphasizes the idea of subdividing a quantity to produce Riemann sums which, in the limit, yield a definite integral. It shows how the integral is used in geometry, physics, economics, and probability; polar coordinates are introduced. It is not necessary to cover all the sections in this chapter.

Chapter 9: Sequences and Series

This chapter focuses on sequences, series of constants, and convergence. It includes the integral, ratio, comparison, limit comparison, and alternating series tests. It also introduces geometric series and general power series, including their intervals of convergence.

Chapter 10: Approximating Functions

This chapter introduces Taylor Series and Fourier Series using the idea of approximating functions by simpler functions.

Chapter 11: Differential Equations

This chapter introduces differential equations. The emphasis is on qualitative solutions, modeling, and interpretation.

Section 11.7 on Logistic Models (formerly on population models) has been rewritten around the thought-provoking predictions of peak oil production. This section encourages students to use the skills learned earlier in the course to analyze a problem of global importance. Sections 11.10 and 11.11 on Second Order Differential Equations are now on the web at www.wiley.com/college/hughes-hallett.

Chapter 12: Functions of Several Variables

This chapter introduces functions of many variables from several points of view, using surface graphs, contour diagrams, and tables. We assume throughout that functions of two or more variables are defined on regions with piecewise smooth boundaries. We conclude with a section on continuity.

Chapter 13: A Fundamental Tool: Vectors

This chapter introduces vectors geometrically and algebraically and discusses the dot and cross product.

Chapter 14: Differentiating Functions of Several Variables

Partial derivatives, directional derivatives, gradients, and local linearity are introduced. The chapter also discusses higher order partial derivatives, quadratic Taylor approximations, and differentiability.

Chapter 15: Optimization

The ideas of the previous chapter are applied to optimization problems, both constrained and unconstrained.

Chapter 16: Integrating Functions of Several Variables

This chapter discusses double and triple integrals in Cartesian, polar, cylindrical, and spherical coordinates.

The former Section 16.7 has been moved to the new Chapter 21.

Chapter 17: Parameterization and Vector Fields

This chapter discusses parameterized curves and motion, vector fields and flowlines.

The former Section 17.5 has been moved to the new Chapter 21.

Chapter 18: Line Integrals

This chapter introduces line integrals and shows how to calculate them using parameterizations. Conservative fields, gradient fields, the Fundamental Theorem of Calculus for Line Integrals, and Green's Theorem are discussed.

Chapter 19: Flux Integrals and Divergence

This chapter introduces flux integrals and shows how to calculate them over surface graphs, portions of cylinders, and portions of spheres. The divergence is introduced and its relationship to flux integrals discussed in the Divergence Theorem.

This new chapter combines Sections 19.1 and 19.2 with Sections 20.1 and 20.2 from the fifth edition

Chapter 20: The Curl and Stokes' Theorem

The purpose of this chapter is to give students a practical understanding of the curl and of Stokes' Theorem and to lay out the relationship between the theorems of vector calculus.

This chapter consists of Sections 20.3–20.5 from the fifth edition.

Chapter 21: Parameters, Coordinates, and Integrals

This new chapter covers parameterized surfaces, the change of variable formula in a double or triple integral, and flux though a parameterized surface.

Appendices

There are appendices on roots, accuracy, and bounds; complex numbers; Newton's Method; and determinants.

Projects

There are new projects in Chapter 12: "Heathrow"; Chapter 19: "Solid Angle"; and Chapter 20: "Magnetic field generated by a current in a wire".

Choice of Paths: Lean or Expanded

For those who prefer the lean topic list of earlier editions, we have kept clear the main conceptual paths. For example,

- The Key Concept chapters on the derivative and the definite integral (Chapters 2 and 5) can be covered at the outset of the course, right after Chapter 1.

- Limits and Continuity (Sections 1.7 and 1.8) can be covered in depth before the introduction of the derivative (Sections 2.1 and 2.2), or after.
- Approximating Functions Using Series (Chapter 10) can be covered before, or without, Chapter 9.
- In Chapter 4 (Using the Derivative), instructors can select freely from Sections 4.3–4.8.
- Chapter 8 (Using the Definite Integral) contains a wide range of applications. Instructors can select one or two to do in detail.

Supplementary Materials and Additional Resources

Supplements for the instructor can be obtained online at the book companion site or by contacting your Wiley representative. The following supplementary materials are available for this edition:

- **Instructor's Manual** containing teaching tips, calculator programs, overhead transparency masters, sample worksheets, and sample syllabi.
- **Computerized Test Bank**, comprised of nearly 7,000 questions, mostly algorithmically-generated, which allows for multiple versions of a single test or quiz.
- **Instructor's Solution Manual** with complete solutions to all problems.
- **Student Solution Manual** with complete solutions to half the odd-numbered problems.
- **Additional Material**, elaborating specially marked points in the text and password-protected electronic versions of the instructor ancillaries, can be found on the web at www.wiley.com/college/hughes-hallett.

ConcepTests

ConcepTests, modeled on the pioneering work of Harvard physicist Eric Mazur, are questions designed to promote active learning during class, particularly (but not exclusively) in large lectures. Our evaluation data show students taught with ConcepTests outperformed students taught by traditional lecture methods 73% versus 17% on conceptual questions, and 63% versus 54% on computational problems.

Faculty Resource Network

A peer-to-peer network of academic faculty dedicated to the effective use of technology in the classroom, this group can help you apply innovative classroom techniques and implement specific software packages. Visit www.facultyresourcenetwork.com or speak to your Wiley representative.

WileyPLUS

WileyPLUS, Wiley's digital learning environment, is loaded with all of the supplements above, and also features:

- Online version of the text, featuring hyperlinks to referenced content, applets, and supplements.
- Homework management tools, which enable the instructor to assign questions easily and grade them automatically, using a rich set of options and controls.
- QuickStart pre-designed reading and homework assignments. Use them as-is or customize them to fit the needs of your classroom.
- Guided Online (GO) Exercises, which prompt students to build solutions step by step. Rather than simply grading an exercise answer as wrong, GO problems show students precisely where they are making a mistake.
- Animated applets, which can be used in class to present and explore key ideas graphically and dynamically—especially useful for display of three-dimensional graphs in multivariable calculus.
- Algebra & Trigonometry Refresher material, which provide students with an opportunity to brush up on material necessary to master Calculus, as well as to determine areas that require further review.
- Graphing Calculator Manual, to help students get the most out of their graphing calculators, and to show how they can apply the numerical and graphing functions of their calculators to their study of calculus.

AP Teacher's Guide

The AP Guide, written by experienced AP teachers, provides day-by-day syllabi for AB and BC Calculus, sample multiple choice questions, a listing of the past 25 years of AP free-response questions by chapter of the text, teaching tips, and labs to encourage student exploration of concepts.

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Daniel E. Flath	Guadalupe I. Lozano	Jeff Tecosky-Feldman
Selin Kalaycıoğlu	Brad G. Osgood	Thomas W. Tucker

To Students: How to Learn from this Book

- This book may be different from other math textbooks that you have used, so it may be helpful to know about some of the differences in advance. This book emphasizes at every stage the *meaning* (in practical, graphical or numerical terms) of the symbols you are using. There is much less emphasis on "plug-and-chug" and using formulas, and much more emphasis on the interpretation of these formulas than you may expect. You will often be asked to explain your ideas in words or to explain an answer using graphs.
- The book contains the main ideas of calculus in plain English. Your success in using this book will depend on your reading, questioning, and thinking hard about the ideas presented. Although you may not have done this with other books, you should plan on reading the text in detail, not just the worked examples.
- There are very few examples in the text that are exactly like the homework problems. This means that you can't just look at a homework problem and search for a similar–looking "worked out" example. Success with the homework will come by grappling with the ideas of calculus.
- Many of the problems that we have included in the book are open-ended. This means that there may be more than one approach and more than one solution, depending on your analysis. Many times, solving a problem relies on common sense ideas that are not stated in the problem but which you will know from everyday life.
- Some problems in this book assume that you have access to a graphing calculator or computer. There are many situations where you may not be able to find an exact solution to a problem, but you can use a calculator or computer to get a reasonable approximation.
- This book attempts to give equal weight to four methods for describing functions: graphical (a picture), numerical (a table of values) algebraic (a formula), and verbal. Sometimes you may find it easier to translate a problem given in one form into another. The best idea is to be flexible about your approach: if one way of looking at a problem doesn't work, try another.
- Students using this book have found discussing these problems in small groups very helpful. There are a great many problems which are not cut-and-dried; it can help to attack them with the other perspectives your colleagues can provide. If group work is not feasible, see if your instructor can organize a discussion session in which additional problems can be worked on.
- You are probably wondering what you'll get from the book. The answer is, if you put in a solid effort, you will get a real understanding of one of the most important accomplishments of the millennium—calculus—as well as a real sense of the power of mathematics in the age of technology.

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Chapter One

A LIBRARY OF FUNCTIONS

Contents

1.1 FUNCTIONS AND CHANGE

In mathematics, a *function* is used to represent the dependence of one quantity upon another.

Let's look at an example. Syracuse, New York has the highest annual snowfall of any US city because of the "lake effect" snow coming from cold Northwest winds blowing over nearby Lake Erie. Lake effect snowfall has been heavier over the last few decades; some have suggested this is due to the warming of Lake Erie by climate change. In December 2010, Syracuse got 66.9 inches of snow in one 12 day period, all of it from lake effect snow. See Table 1.1.

Table 1.1 *Daily snowfall in Syracuse, December 5–16, 2010*

Date (December 2010)	5	6	7	8	9	10	11	12	13	14	15	16
Snowfall in inches	6.8	12.2	9.3	14.9	1.9	0.1	0.0	0.0	1.4	5.0	11.9	3.4

You may not have thought of something so unpredictable as daily snowfall as being a function, but it *is* a function of date, because each day gives rise to one snowfall total. There is no formula for the daily snowfall (otherwise we would not need a weather bureau), but nevertheless the daily snowfall in Syracuse does satisfy the definition of a function: Each date, t, has a unique snowfall, S, associated with it.

We define a function as follows:

> A **function** is a rule that takes certain numbers as inputs and assigns to each a definite output number. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

The input is called the *independent variable* and the output is called the *dependent variable*. In the snowfall example, the domain is the set of December dates $\{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ and the range is the set of daily snowfalls $\{0.0, 0.1, 1.4, 1.9, 3.4, 5.0, 6.8, 9.3, 11.9, 12.2, 14.9\}$. We call the function f and write $S = f(t)$. Notice that a function may have identical outputs for different inputs (December 11 and 12, for example).

Some quantities, such as date, are *discrete*, meaning they take only certain isolated values (dates must be integers). Other quantities, such as time, are *continuous* as they can be any number. For a continuous variable, domains and ranges are often written using interval notation:

The set of numbers t such that $a \le t \le b$ is called a *closed interval* and written $[a, b]$.

The set of numbers t such that $a < t < b$ is called an *open interval* and written (a, b).

The Rule of Four: Tables, Graphs, Formulas, and Words

Functions can be represented by tables, graphs, formulas, and descriptions in words. For example, the function giving the daily snowfall in Syracuse can be represented by the graph in Figure 1.1, as well as by Table 1.1.

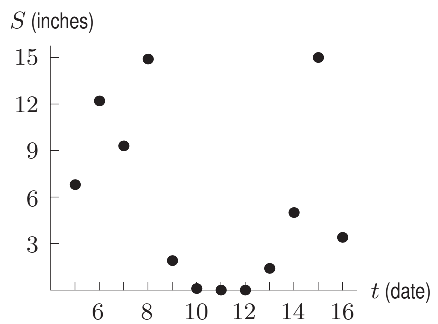

Figure 1.1: Syracuse snowfall, December, 2010

As another example of a function, consider the snow tree cricket. Surprisingly enough, all such crickets chirp at essentially the same rate if they are at the same temperature. That means that the chirp rate is a function of temperature. In other words, if we know the temperature, we can determine the chirp rate. Even more surprisingly, the chirp rate, C, in chirps per minute, increases steadily with the temperature, T, in degrees Fahrenheit, and can be computed by the formula

$$C = 4T - 160$$

to a fair degree of accuracy. We write $C = f(T)$ to express the fact that we think of C as a function of T and that we have named this function f. The graph of this function is in Figure 1.2.

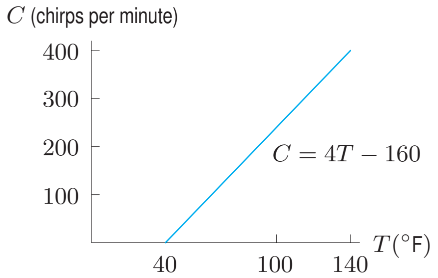

Figure 1.2: Cricket chirp rate versus temperature

Examples of Domain and Range

If the domain of a function is not specified, we usually take it to be the largest possible set of real numbers. For example, we usually think of the domain of the function $f(x) = x^2$ as all real numbers. However, the domain of the function $g(x) = 1/x$ is all real numbers except zero, since we cannot divide by zero.

Sometimes we restrict the domain to be smaller than the largest possible set of real numbers. For example, if the function $f(x) = x^2$ is used to represent the area of a square of side x, we restrict the domain to nonnegative values of x.

Example 1 The function $C = f(T)$ gives chirp rate as a function of temperature. We restrict this function to temperatures for which the predicted chirp rate is positive, and up to the highest temperature ever recorded at a weather station, 136°F. What is the domain of this function f?

Solution If we consider the equation

$$C = 4T - 160$$

simply as a mathematical relationship between two variables C and T, any T value is possible. However, if we think of it as a relationship between cricket chirps and temperature, then C cannot be less than 0. Since $C = 0$ leads to $0 = 4T - 160$, and so $T = 40$°F, we see that T cannot be less than 40°F. (See Figure 1.2.) In addition, we are told that the function is not defined for temperatures above 136°. Thus, for the function $C = f(T)$ we have

$$\begin{aligned}\text{Domain} &= \text{All } T \text{ values between } 40°\text{F and } 136°\text{F}\\ &= \text{All } T \text{ values with } 40 \le T \le 136\\ &= [40, 136].\end{aligned}$$

Example 2 Find the range of the function f, given the domain from Example 1. In other words, find all possible values of the chirp rate, C, in the equation $C = f(T)$.

Solution Again, if we consider $C = 4T - 160$ simply as a mathematical relationship, its range is all real C values. However, when thinking of the meaning of $C = f(T)$ for crickets, we see that the function predicts cricket chirps per minute between 0 (at $T = 40$°F) and 384 (at $T = 136$°F). Hence,

$$\begin{aligned}\text{Range} &= \text{All } C \text{ values from } 0 \text{ to } 384\\ &= \text{All } C \text{ values with } 0 \le C \le 384\\ &= [0, 384].\end{aligned}$$

In using the temperature to predict the chirp rate, we thought of the temperature as the *independent variable* and the chirp rate as the *dependent variable*. However, we could do this backward, and calculate the temperature from the chirp rate. From this point of view, the temperature is dependent on the chirp rate. Thus, which variable is dependent and which is independent may depend on your viewpoint.

Linear Functions

The chirp-rate function, $C = f(T)$, is an example of a *linear function*. A function is linear if its slope, or rate of change, is the same at every point. The rate of change of a function that is not linear may vary from point to point.

Olympic and World Records

During the early years of the Olympics, the height of the men's winning pole vault increased approximately 8 inches every four years. Table 1.2 shows that the height started at 130 inches in 1900, and increased by the equivalent of 2 inches a year. So the height was a linear function of time from 1900 to 1912. If y is the winning height in inches and t is the number of years since 1900, we can write

$$y = f(t) = 130 + 2t.$$

Since $y = f(t)$ increases with t, we say that f is an *increasing function.* The coefficient 2 tells us the rate, in inches per year, at which the height increases.

Table 1.2 *Men's Olympic pole vault winning height (approximate)*

Year	1900	1904	1908	1912
Height (inches)	130	138	146	154

This rate of increase is the *slope* of the line in Figure 1.3. The slope is given by the ratio

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{146 - 138}{8 - 4} = \frac{8}{4} = 2 \text{ inches/year}.$$

Calculating the slope (rise/run) using any other two points on the line gives the same value.

What about the constant 130? This represents the initial height in 1900, when $t = 0$. Geometrically, 130 is the *intercept* on the vertical axis.

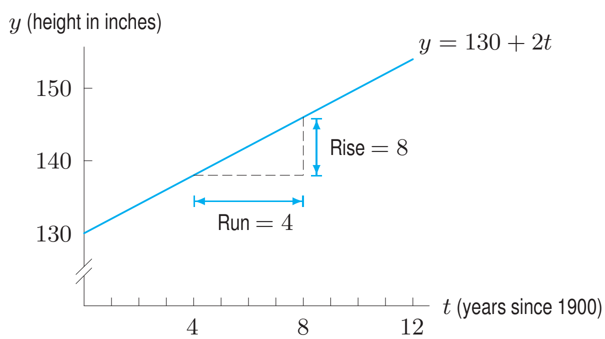

Figure 1.3: Olympic pole vault records

You may wonder whether the linear trend continues beyond 1912. Not surprisingly, it doesn't exactly. The formula $y = 130+2t$ predicts that the height in the 2008 Olympics would be 346 inches or 28 feet 10 inches, which is considerably higher than the actual value of 19 feet 6.65 inches. There is clearly a danger in *extrapolating* too far from the given data. You should also observe that the data in Table 1.2 is discrete, because it is given only at specific points (every four years). However, we have treated the variable t as though it were continuous, because the function $y = 130 + 2t$ makes sense for all values of t. The graph in Figure 1.3 is of the continuous function because it is a solid line, rather than four separate points representing the years in which the Olympics were held.

As the pole vault heights have increased over the years, the time to run the mile has decreased. If y is the world record time to run the mile, in seconds, and t is the number of years since 1900, then records show that, approximately,

$$y = g(t) = 260 - 0.39t.$$

The 260 tells us that the world record was 260 seconds in 1900 (at $t = 0$). The slope, -0.39, tells us that the world record decreased by about 0.39 seconds per year. We say that g is a *decreasing function*.

Difference Quotients and Delta Notation

We use the symbol Δ (the Greek letter capital delta) to mean "change in," so Δx means change in x and Δy means change in y.

The slope of a linear function $y = f(x)$ can be calculated from values of the function at two points, given by x_1 and x_2, using the formula

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The quantity $(f(x_2) - f(x_1))/(x_2 - x_1)$ is called a *difference quotient* because it is the quotient of two differences. (See Figure 1.4.) Since $m = \Delta y/\Delta x$, the units of m are y-units over x-units.

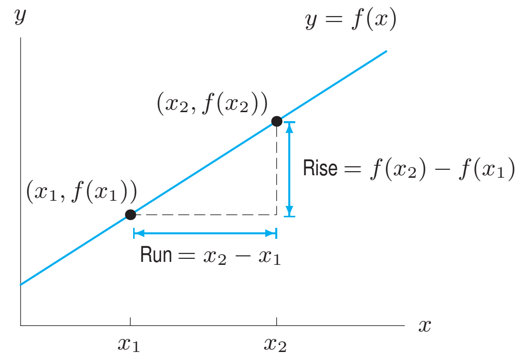

Figure 1.4: Difference quotient $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Families of Linear Functions

A **linear function** has the form

$$y = f(x) = b + mx.$$

Its graph is a line such that

- m is the **slope**, or rate of change of y with respect to x.
- b is the **vertical intercept**, or value of y when x is zero.

Notice that if the slope, m, is zero, we have $y = b$, a horizontal line.

To recognize that a table of x and y values comes from a linear function, $y = b + mx$, look for differences in y-values that are constant for equally spaced x-values.

Formulas such as $f(x) = b + mx$, in which the constants m and b can take on various values, give a *family of functions*. All the functions in a family share certain properties—in this case, all the